Math 4550 - Homework # 6 -
Normal subgroups and Factor Groups

Part 1 - Computations

1. For the following groups G and subgroups H compute the left and right cosets. Are they equal? Is H a normal subgroup? Draw a picture of how the cosets partition the group G.

   (a) $G = \mathbb{Z}_{12}$ and $H = \langle \overline{4} \rangle$.

   (b) $G = D_8$ and $H = \langle r^2 \rangle$.

   (c) $G = D_8$ and $H = \langle s \rangle$.

2. Let $G = \mathbb{Z}_{12}$ and $H = \langle \overline{4} \rangle$.

   H is a normal subgroup of G. You don't have to check this fact.

   (a) Calculate the elements of G/H. What is the identity element of G/H?

   (b) Calculate $(\overline{2} + H) + (\overline{3} + H)$. Find the inverse of $\overline{3} + H$.

   (c) Find the order of $\overline{1} + H$ in G/H. Find the order of $\overline{2} + H$ in G/H.

   (d) Is G/H abelian? Is G/H cyclic? If G/H is cyclic, state a generator.

3. Let $G = D_8$ and $H = \langle s \rangle$.

   H is a normal subgroup of G. You don't have to check this fact.

   (a) Calculate the elements of G/H. What is the identity element of G/H?

   (b) Calculate $(rH)(sH)$ and $(srH)(srH)$.

   (c) Find the inverse of rH. Find the inverse of sH.

   (d) Find the orders of H, rH, sH, and srH in G/H.

   (e) Is G/H cyclic? If G/H is cyclic, state a generator.

4. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ and $H = \langle (\overline{0}, \overline{2}) \rangle$.

   H is a normal subgroup of G. You don't have to check this fact.

   (a) Calculate the elements of G/H. What is the identity element of G/H?

   (b) Calculate $[(\overline{1}, \overline{2}) + H] + [(\overline{1}, \overline{1}) + H]$ and $[(\overline{0}, \overline{1}) + H] + [(\overline{2}, \overline{1}) + H]$.

   (c) Find the inverse of $(\overline{1}, \overline{2}) + H$. Find the inverse of $(\overline{2}, \overline{2}) + H$.

   (d) Find the orders all the elements in the group.

   (e) Show that G/H cyclic and list all of it's generators.

Part 2 - Proofs

5. Let G be a group and $H \leq G$. Let $a, b \in G$.
   (a) Prove that $aH = bH$ iff $a \in bH$.
   (b) Prove that $aH = bH$ iff $a = bh$ for some $h \in H$.
   (c) Show that if $|H|$ is finite, then $|H| = |aH|$.
   [Hint: Show that $f : H \to aH$ where $f(h) = ah$ is one-to-one and onto.]

6. Let G be a group where $|G| = pq$ where p and q are primes and $p \neq q$.
   Let H be a subgroup of G with $H \neq G$. Prove that H is cyclic.

7. Let G be a group with identity element e. Suppose that $|G| = n$. Prove that $x^n = e$ for all $x \in G$.

8. Let G be a group and H be a normal subgroup of G.
   (a) Prove that if G is a abelian, then G/H is abelian.
   (b) Show that $(aH)^{-1} = (a^{-1})H$ for any $a \in G$.
   (c) Show that $(aH)^k = (a^k)H$ for any $a \in G$ and integer k.
   (d) Prove that if G is a cyclic, then G/H is cyclic.

9. Let G be a group and H and K be normal subgroups of G. Prove that $H \cap K$ is a normal subgroup of G.

10. Let G_1 and G_2 be groups. Let $\phi : G_1 \to G_2$ be a homomorphism. Show that the kernel of ϕ is a normal subgroup of G_1.

Part 3 - The problem below is OPTIONAL.

11. Let G be a finite group and H be a subgroup of G. Prove that if H is only subgroup of G of size $|H|$, then H is normal in G.

    [*Hint: Given $g \in G$, consider $\phi_g : H \to G$ given by $\phi_g(h) = ghg^{-1}$. Use the fact that the image of a homomorphism is a subgroup.*]